CS103
Winter 2025

Lecture 17:

Regular Expressions

First, Some Announcements!

Second Midterm Logistics

- Our second midterm is next ***Tuesday, February 25th,*** from ***7-9 PM***. Locations vary, but mostly CEMEX.
- Topic coverage is primarily lectures 06 – 13 (functions through induction) and PS3 – PS5. Finite automata and onward won't be tested here.
  - Because the material is cumulative, topics from PS1 – PS2 and Lectures 00 – 05 are also fair game.
- Seating assignments are posted.
- Anisha and Zach will host an exam review session this Sunday, February 23rd, 4-6 PM, in CoDa E160.

Preparing for the Exam

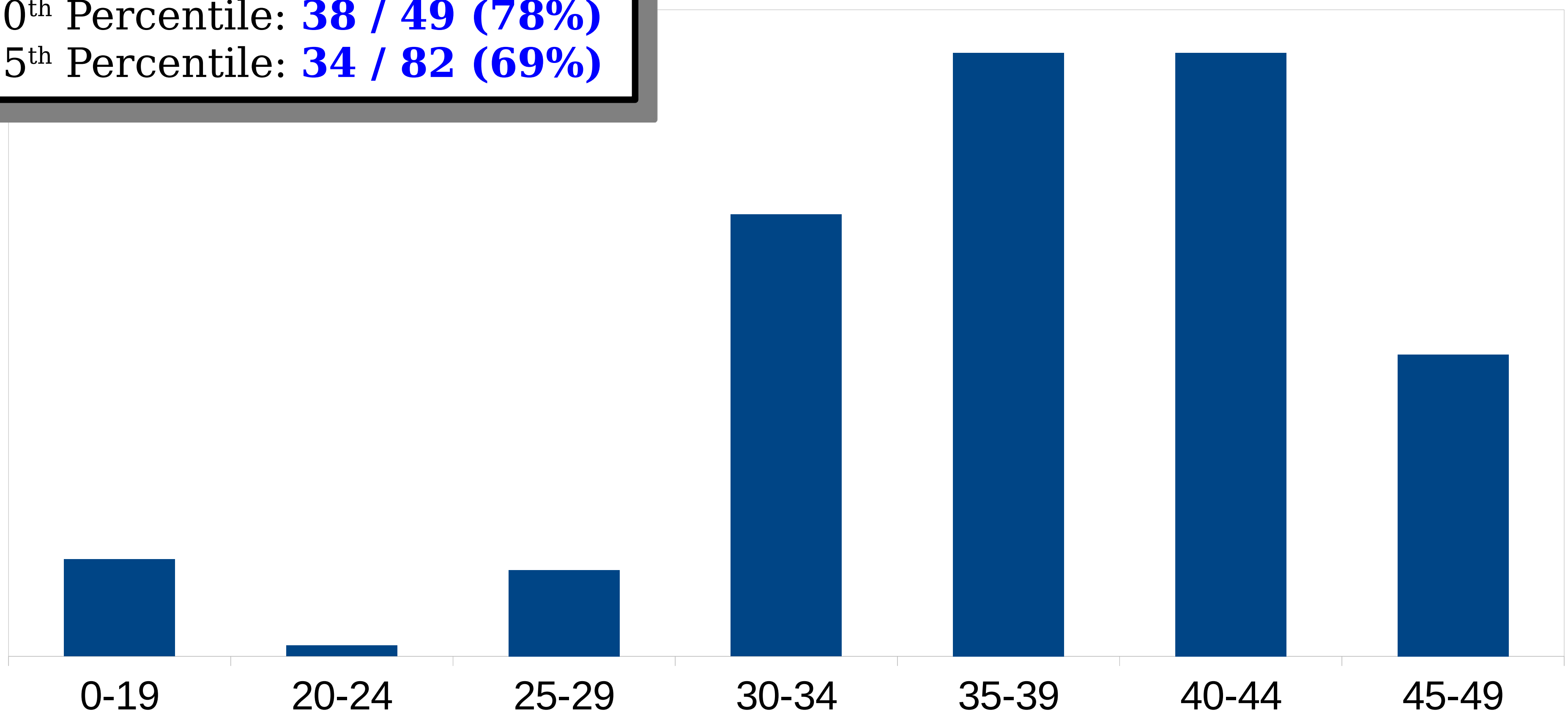
- The top skills that will serve you well on this exam:
  - ***Knowing how to set up a proof***. This is a recurring theme across functions, sets, graphs, pigeonhole, and induction.
  - ***Distinguishing between assuming and proving***. This similarly cuts across all of these topics.
  - ***Reading new definitions***. This is at the heart of mathematical reasoning.
  - ***Writing proofs in line with definitions***. Folks often ask about whether they're being rigorous enough. Often "rigorous enough" simply means "following what the definitions say."
- Our personal recommendation: when working through practice problems, pay super extra close attention to these areas.

Preparing for the Exam

- As with the first midterm exam, we’ve posted a bunch of practice exams on the course website.
  - There are ten practice exams (yes, really!). We realistically don’t expect anyone to complete them all. They’re there to give you a feeling of what the exam might look like.
- Some general notes on preparing:
  - Q5 and Q6 on PS6, while technically on topics that aren’t covered on the midterm, are great practice for the sorts of reasoning you’ll need on the exam.
  - ***Keep the TAs in the loop when studying***. Ask for feedback on any proofs you write when getting ready for the exam.
  - Don’t skip on biological care and maintenance. Exams can be stressful, but please make time for basic things like showering, eating, etc. and for self-care in whatever form that takes for you.
- ***You can do this***. Best of luck on the exam!

Problem Set Five Graded

75th Percentile: **44 / 49 (90%)**
50th Percentile: **38 / 49 (78%)**
25th Percentile: **34 / 82 (69%)**



On to CS103!

Recap from Last Time

Regular Languages

- A language L is called a ***regular language*** if there is a DFA or an NFA for L.
- ***Theorem:*** The following are equivalent:
  - L is a regular language.
  - There is a DFA D where $\mathscr{L}(D) = L$.
  - There is an NFA N where $\mathscr{L}(N) = L$.
- In other words, knowing any one of the above three facts means you know the other two.

Language Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, then wx is the ***concatenation*** of w and x.
- If L_1 and L_2 are languages over Σ, the ***concatenation*** of L_1 and L_2 is the language L_1L_2 defined as

$$L_1L_2 = \{ x \mid \exists w_1 \in L_1.\ \exists w_2 \in L_2.\ x = w_1w_2 \}$$

- Example: if $L_1 = \{$ **a**, **ba**, **bb** $\}$ and $L_2 = \{$ **aa**, **bb** $\}$, then

$L_1L_2 = \{$ **aaa**, **abb**, **baaa**, **babb**, **bbaa**, **bbbb** $\}$

Lots and Lots of Concatenation

- Consider the language L = { aa, b }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ aaaa, aab, baa, bb }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ aaaaaa, aaaab, aabaa, aabb, baaaa, baab, bbaa, bbb}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ aaaaaaaa, aaaaaab, aaaabaa, aaaabb, aabaaaa, aabaab, aabbaa, aabbb, baaaaaa, baaaab, baabaa, baabb, bbaaaa, bbaab, bbbaa, bbbb}

Language Exponentiation

- We can define what it means to "exponentiate" a language as follows:

$$L^0 = \{\varepsilon\} \qquad\qquad L^{n+1} = LL^n$$

- So, for example, { **aa**, **b** }^{3} is the language

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**,
baaaa, **baab**, **bbaa**, **bbb**}

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$\boldsymbol{L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}}$$

- Mathematically:

$$\boldsymbol{w \in L^* \quad \text{iff} \quad \exists n \in \mathbb{N}.\ w \in L^n}$$

- Intuitively, all possible ways of concatenating zero or more strings in L together, possibly with repetition.

The Kleene Closure

If L = { a, bb }, then L^* = {

ε,

a, bb,

aa, abb, bba, bbbb,

aaa, aabb, abba, abbbb, bbaa, bbabb, bbbba, bbbbbb,

…

}

Think of L* as the set of strings you can make if you have a collection of rubber stamps – one for each string in L – and you form every possible string that can be made from those stamps.

Closure Properties

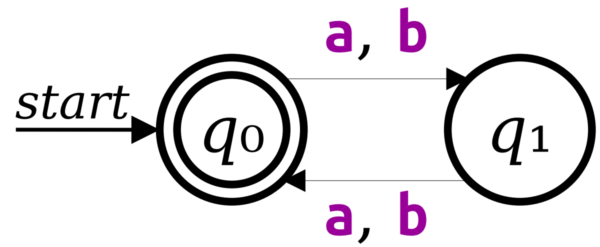
- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $L_1 \cup L_2$
  - L_1L_2
  - L_1^*
- These (and other) properties are called ***closure properties of the regular languages***.

New Stuff!

Another View of Regular Languages

Devices for Articulating Regular Languages

- Finite Automata



- Set (or other Mathematical) Notation

{ $w \in \Sigma^*$ | w’s length is even }

- State Transition Table

	a	**b**
q_0	q_1	q_1
q_1	q_0	q_0

- ***New!*** Regular Expressions

Devices for Articulating Regular Languages

- Finite Automata

- Set (or other Mathematical) Notation

$$\{\ w \in \Sigma^* \mid w\text{'s length is even}\ \}$$

- State Transition

- ***New!*** Regular

Note: This one is not unique to regular languages! We can express non-regular languages with set builder notation, as well. More on that another day, when we explore other families of languages.

Regular Expressions

- ***Regular expressions*** are a way of describing a language via a string representation.
- They're used just about everywhere:
  - They're built into the JavaScript language and used for data validation.
  - They're used in the UNIX `grep` and `flex` tools to search files and build compilers.
  - They're employed to clean and scrape data for large-scale analysis projects.
- Conceptually, regular expressions are strings describing how to assemble a larger language out of smaller pieces.

Rethinking Regular Languages

- We currently have several tools for showing a language L is regular:
  - Construct a DFA for L.
  - Construct an NFA for L.
  - Combine several simpler regular languages together via closure properties to form L.
- We have not spoken much of this last idea.

Constructing Regular Languages

- ***Idea:*** Build up all regular languages as follows:
  - Start with a small set of simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more elaborate languages.
- *This is a bottom-up approach to the regular languages.*

Atomic Regular Expressions

- The regular expressions begin with three simple building blocks.
- The symbol **Ø** is a regular expression that represents the empty language Ø.
- For any **a** ∈ Σ, the symbol **a** is a regular expression for the language {**a**}.
- The symbol **ε** is a regular expression that represents the language {ε}.
  - ***Remember:* {ε} ≠ Ø!**
  - ***Remember:* {ε} ≠ ε!**

Compound Regular Expressions

- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1R_2}$ is a regular expression for the *concatenation* of the languages of R_1 and R_2.
- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1 \cup R_2}$ is a regular expression for the *union* of the languages of R_1 and R_2.
- If R is a regular expression, $\boldsymbol{R^*}$ is a regular expression for the *Kleene closure* of the language of R.
- If R is a regular expression, $\boldsymbol{(R)}$ is a regular expression with the same meaning as R.

Operator Precedence

- Here's the operator precedence for regular expressions:

$$(R)$$

$$R^*$$

$$R_1R_2$$

$$R_1 \cup R_2$$

- So **ab*c∪d** is parsed as **((a(b*))c)∪d**

Regular Expression Examples

- The regular expression **trick∪treat** represents the language

  { **trick**, **treat** }.

- The regular expression **booo*** represents the regular language

  { **boo**, **booo**, **boooo**, … }.

- The regular expression **candy!(candy!)*** represents the regular language

  { **candy!**, **candy!candy!**, **candy!candy!candy!**, … }.

Regular Expressions, Formally

- The ***language of a regular expression*** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\boldsymbol{\varepsilon}) = \{\varepsilon\}$
  - $\mathscr{L}(\boldsymbol{\emptyset}) = \emptyset$
  - $\mathscr{L}(\mathbf{a}) = \{\mathbf{a}\}$
  - $\mathscr{L}(R_1R_2) = \mathscr{L}(R_1)\ \mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \cup R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R\boldsymbol{*}) = \mathscr{L}(R)^*$
  - $\mathscr{L}(\boldsymbol{(}R\boldsymbol{)}) = \mathscr{L}(R)$

Worthwhile activity: Apply this recursive definition to

a(b∪c)((d))

and see what you get.

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { $w \in \Sigma^*$ | w contains **aa** as a substring }.

(a ∪ b)*aa(a ∪ b)*

bbabbbaabab
aaaa
bbbbbabbbbaabbbbb

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

Σ*aaΣ*

bbabbbaabab
aaaa
bbbbbabbbbaabbbbb

Designing Regular Expressions

Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.

Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

The length of a string w is denoted $|w|$

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$\Sigma\Sigma\Sigma\Sigma$

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Σ^4

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { w ∈ Σ* | w contains at most one **a** }.

Here are some candidate regular expressions for the language L. Which of these are correct?

Σ*aΣ*
b*ab* ∪ b*
b*(a ∪ ε)b*
b*a*b* ∪ b*
b*(a* ∪ ε)b*

Answer at ***https://cs103.stanford.edu/pollev***

Designing Regular Expressions

- Let $\Sigma = \{$a, b$\}$.
- Let $L = \{\ w \in \Sigma^* \mid w$ contains at most one a $\}$.

b*(a ∪ ε)b*

bbbbabbb
bbbbbb
abbb
a

Designing Regular Expressions

- Let Σ = {`a`, `b`}.
- Let L = { $w \in \Sigma^*$ | w contains at most one `a` }.

b*a?b*

bbbbabbb
bbbbbb
abbb
a

A More Elaborate Design

- Let Σ = { `a`, `.`, `@` }, where `a` represents "some letter."
- Let's make a regex for email addresses.

`aa* (.aa*)* @ aa*.aa* (.aa*)*`

`cs103@cs.stanford.edu`
`first.middle.last@mail.site.org`
`dot.at@dot.com`

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$\mathtt{a}^+ \; (\mathtt{.a}^+)^* \; \mathtt{@} \; \mathtt{a}^+ \, \mathtt{.a}^+ \; (\mathtt{.a}^+)^*$$

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

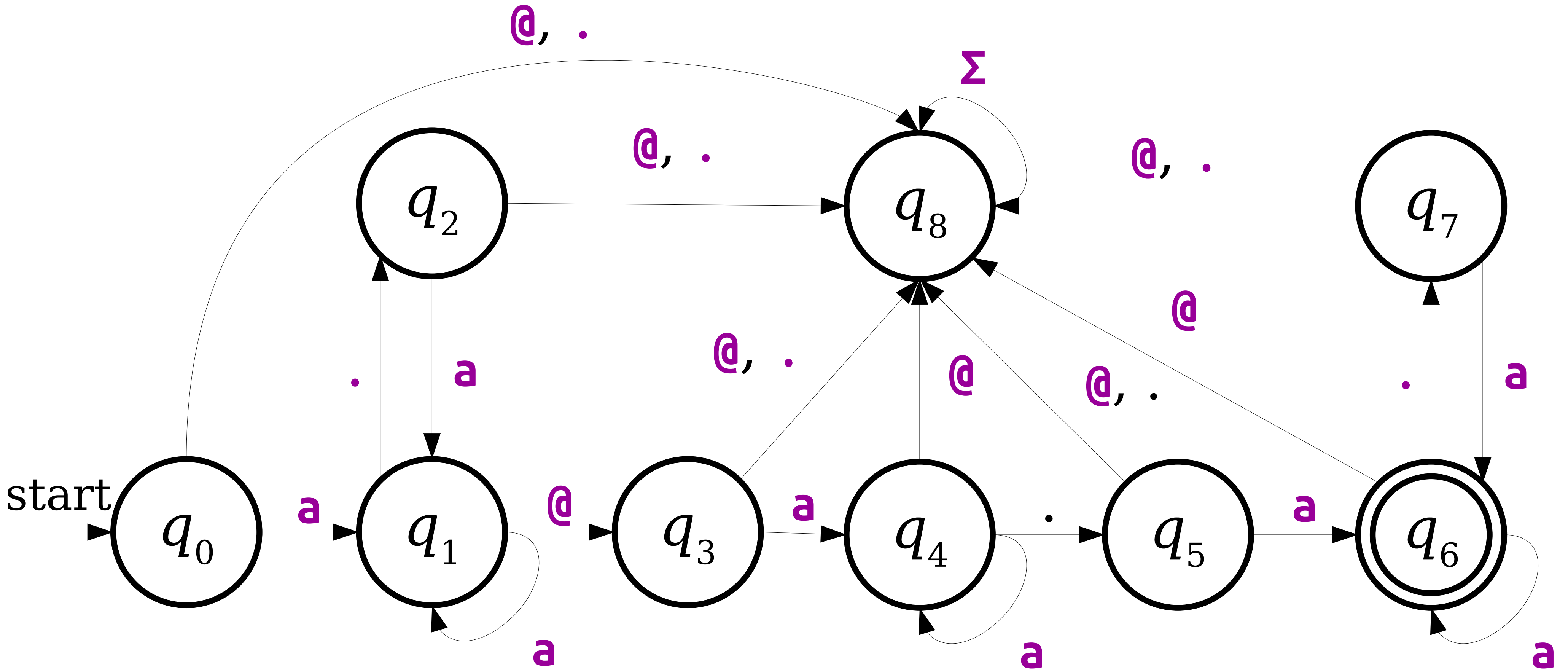
- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$\mathtt{a^{+}\,(.a^{+})^{*}\,@\,a^{+}(.a^{+})^{+}}$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

For Comparison

$\mathtt{a^+(.a^+)^*@a^+(.a^+)^+}$



Shorthand Summary

- $\mathbf{R}^n$ is shorthand for **RR** … **R** (n times).
  - Edge case: define $\mathbf{R}^0$ = ε.
- **Σ** is shorthand for "any character in Σ."
- **R?** is shorthand for **(R ∪ ε)**, meaning "zero or one copies of R."
- $\mathbf{R}^+$ is shorthand for **RR***, meaning "one or more copies of R."

The Lay of the Land

The Power of Regular Expressions

Theorem: If R is a regular expression, then $\mathscr{L}(R)$ is regular.

Proof idea: Use induction!

- The atomic regular expressions all represent regular languages.
- The combination steps represent closure properties.
- So anything you can make from them must be regular!

Thompson’s Algorithm

- In practice, many regex matchers use an algorithm called ***Thompson's algorithm*** to convert regular expressions into NFAs (and, from there, to DFAs).
  - Read Sipser if you’re curious!
- ***Fun fact:*** the “Thompson” here is Ken Thompson, one of the co-inventors of Unix!

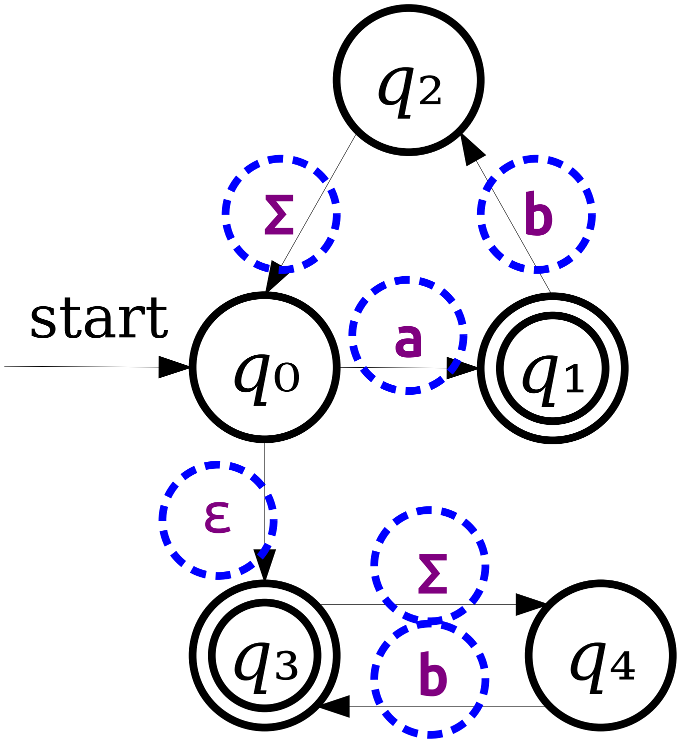
The Power of Regular Expressions

Theorem: If L is a regular language, then there is a regular expression for L.

This is not obvious!

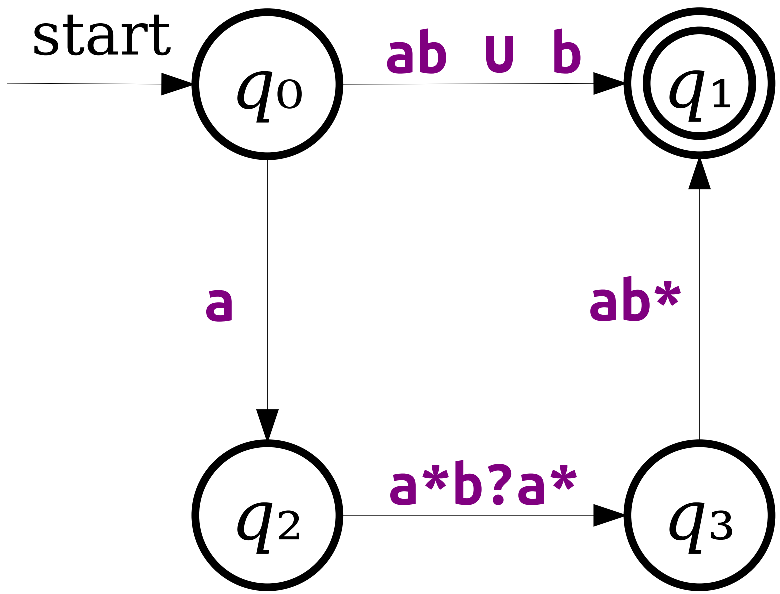
Proof idea: Show how to convert an arbitrary NFA into a regular expression.

Generalizing NFAs



These are all regular expressions!

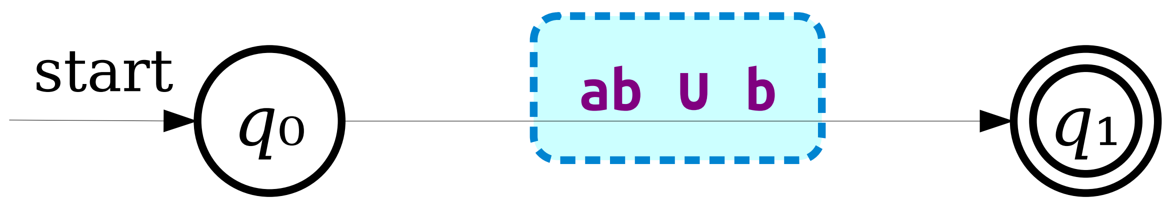
Generalizing NFAs



Note: Actual NFAs aren't allowed to have transitions like these. This is just a thought experiment.

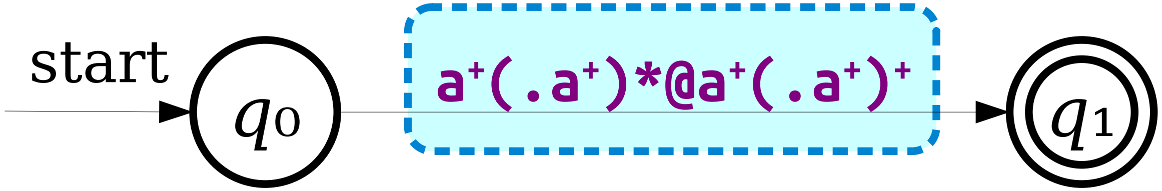
Key Idea 1: Imagine that we can label transitions in an NFA with arbitrary regular expressions.

Generalizing NFAs



Is there a simple regular expression for the language of this generalized NFA?

Generalizing NFAs

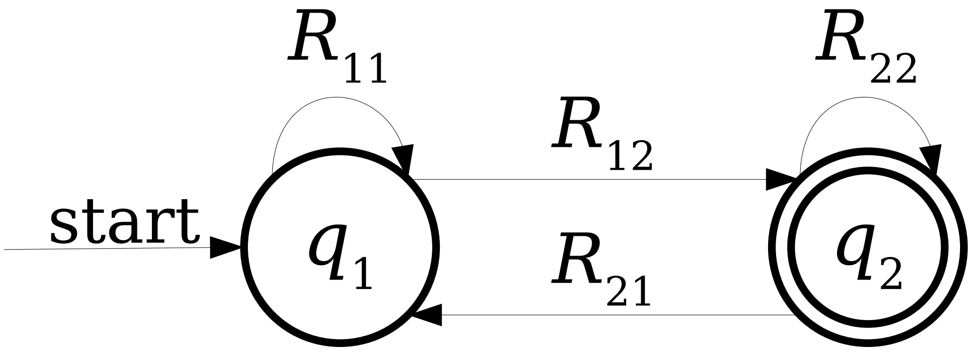


Is there a simple regular expression for the language of this generalized NFA?

Key Idea 2: If we can convert an NFA into a generalized NFA that looks like this…

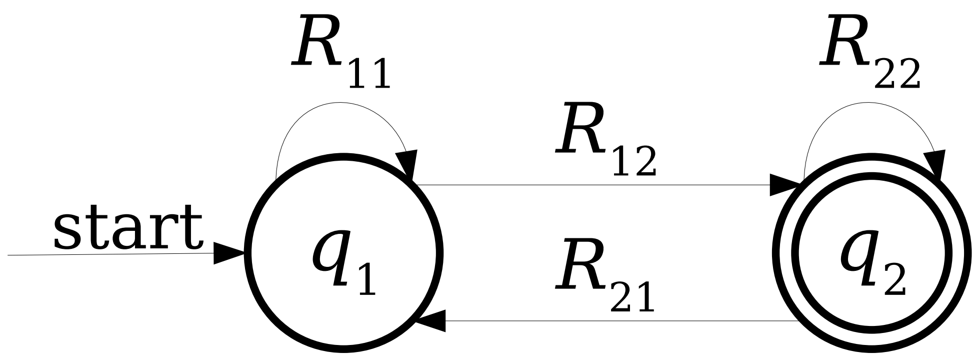
…then we can easily read off a regular expression for the original NFA.

From NFAs to Regular Expressions



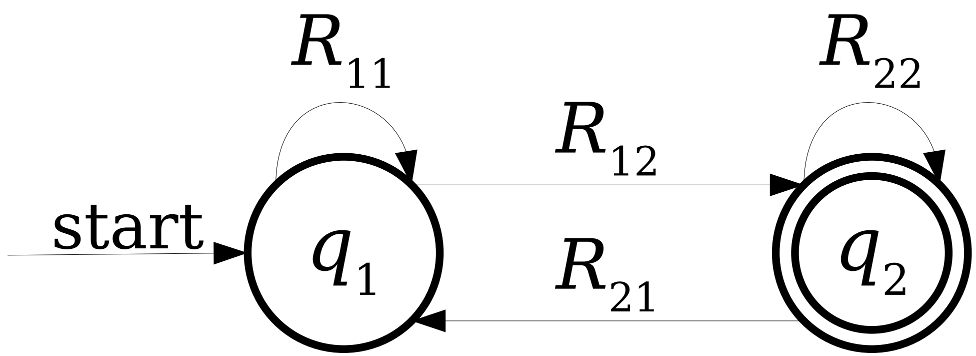
Here, R_{11}, R_{12}, R_{21}, and R_{22} are arbitrary regular expressions.

From NFAs to Regular Expressions



Question: Can we get a clean regular expression from this NFA?

From NFAs to Regular Expressions



Key Idea 3: Somehow transform this NFA so that it looks like this:

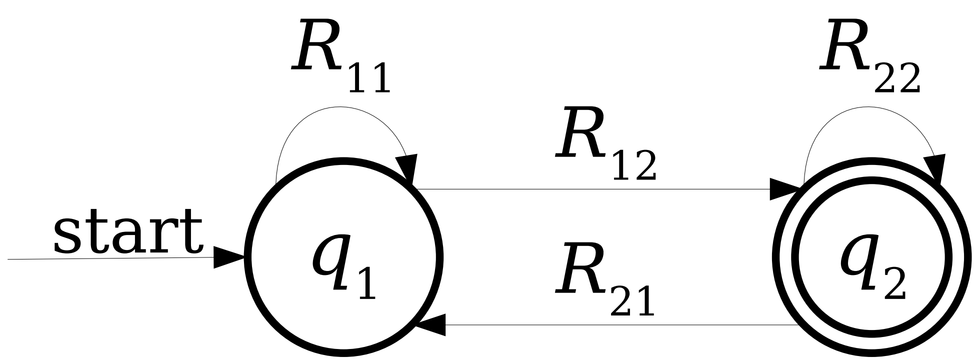
The State-Elimination Algorithm

- Start with an NFA N for the language L.
- Add a new start state q_s and accept state q_f to the NFA.
  - Add an ε-transition from q_s to the old start state of N.
  - Add ε-transitions from each accepting state of N to q_f, then mark them as not accepting.
- Repeatedly remove states other than q_s and q_f from the NFA by "shortcutting" them until only two states remain: q_s and q_f.
- The transition from q_s to q_f is then a regular expression for the NFA.

The State-Elimination Algorithm

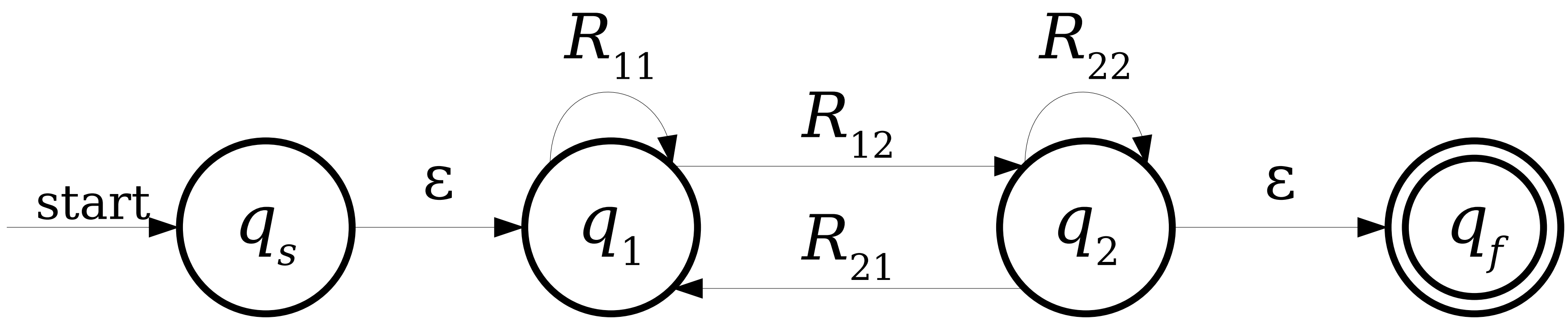
- To eliminate a state q from the automaton, do the following for each pair of states q_0 and q_1, where there's a transition from q_0 into q and a transition from q into q_1:
  - Let R_{in} be the regex on the transition from q_0 to q.
  - Let R_{out} be the regex on the transition from q to q_1.
  - If there is a regular expression R_{stay} on a transition from q to itself, add a new transition from q_0 to q_1 labeled $((R_{in})(R_{stay})^*(R_{out}))$.
  - If there isn't, add a new transition from q_0 to q_1 labeled $((R_{in})(R_{out}))$
- If a pair of states has multiple transitions between them labeled R_1, R_2, …, R_k, replace them with a single transition labeled $R_1 \cup R_2 \cup \ldots \cup R_k$.

From NFAs to Regular Expressions

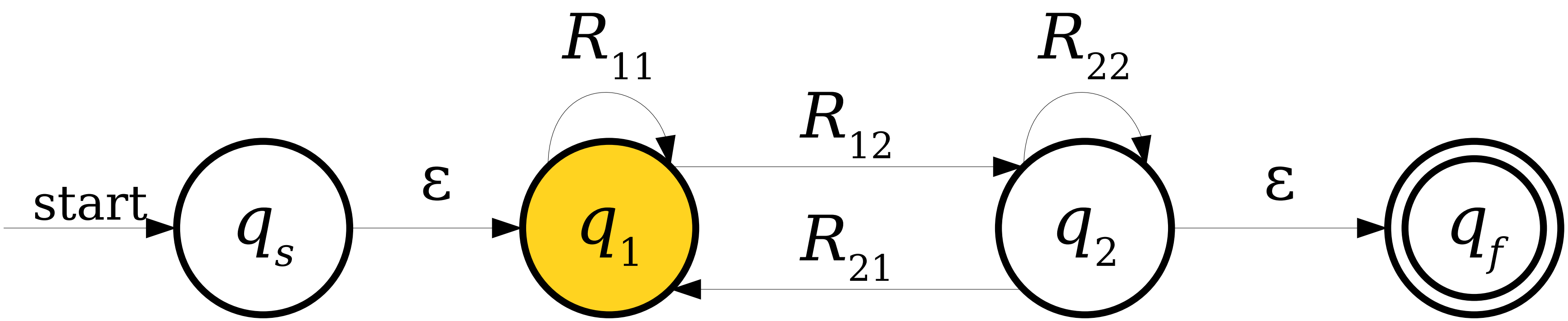


The first step is going to be a bit weird...

From NFAs to Regular Expressions

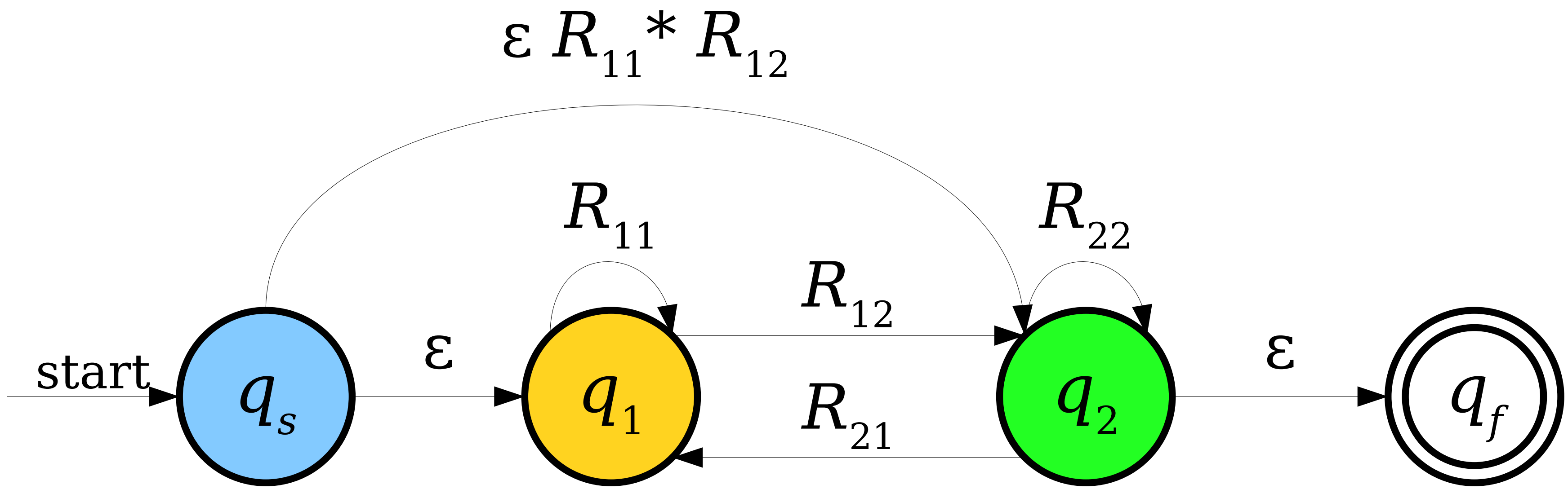


From NFAs to Regular Expressions



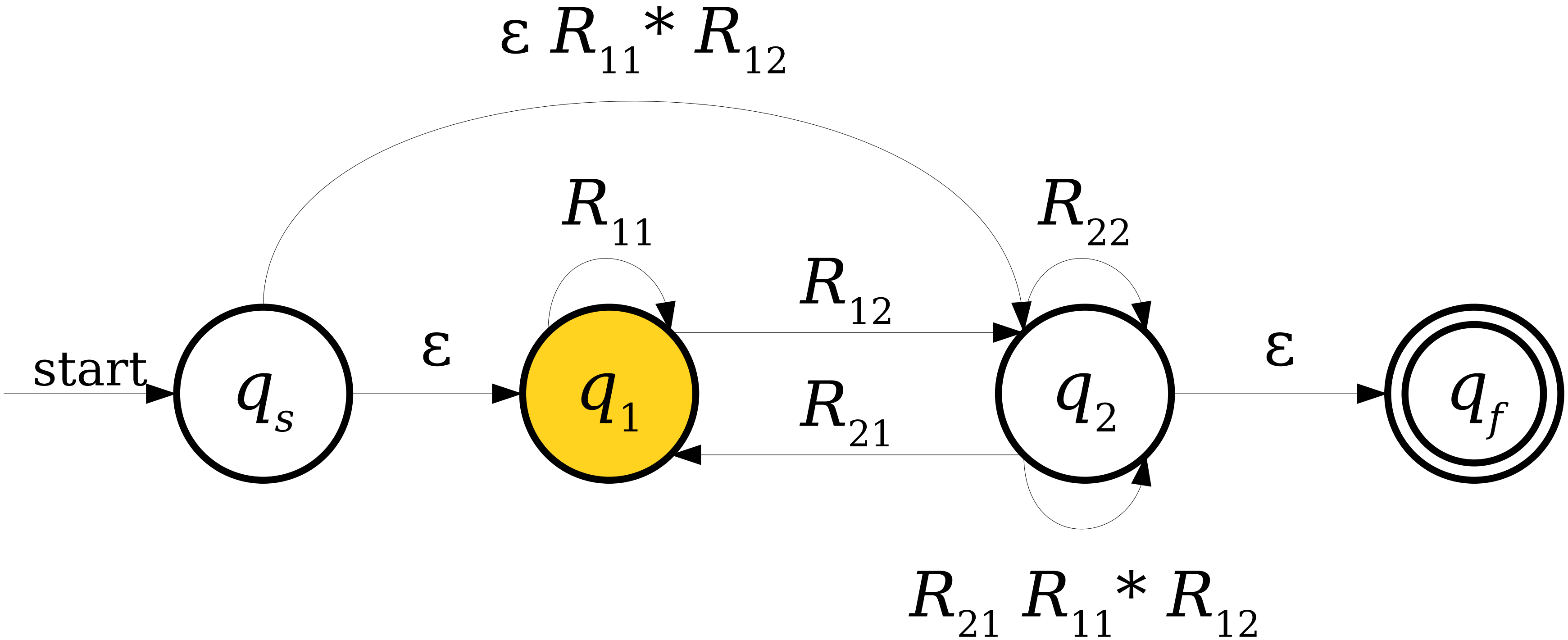
Could we eliminate this state from the NFA?

From NFAs to Regular Expressions

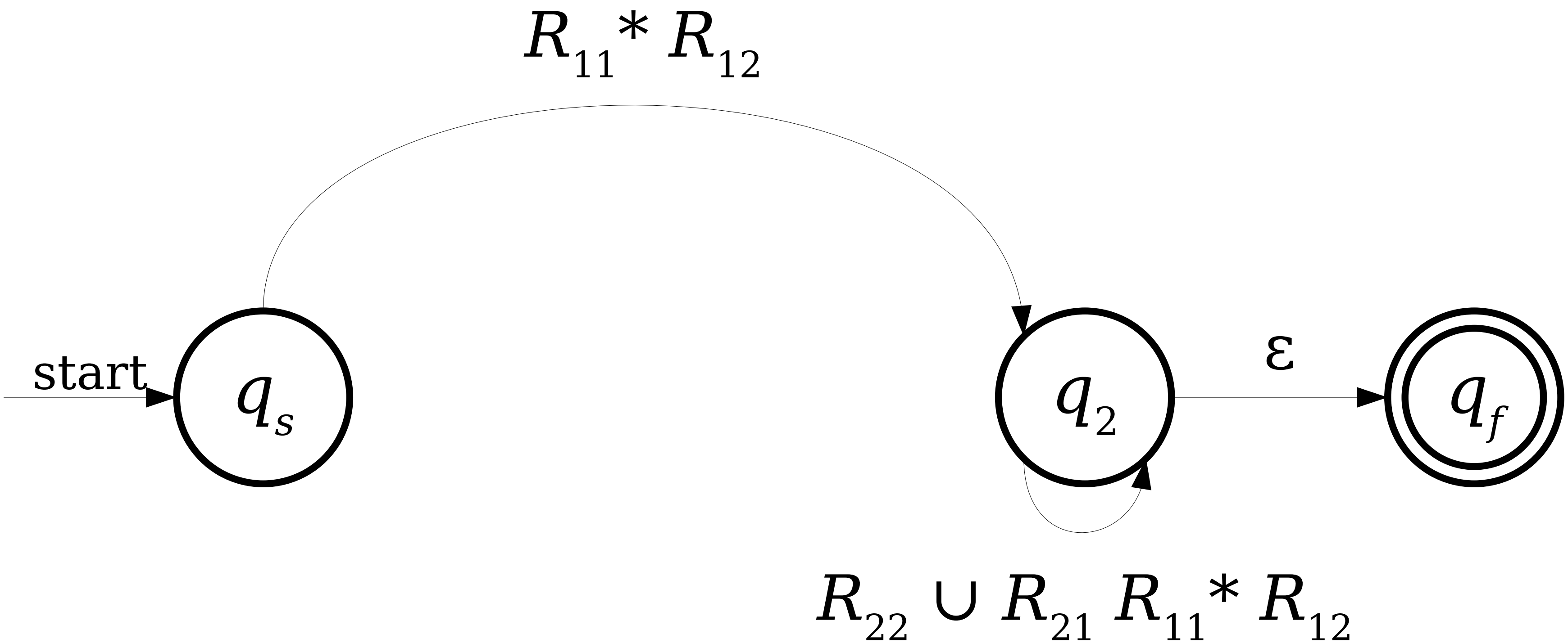


Note: We're using concatenation and Kleene closure in order to skip this state.

From NFAs to Regular Expressions

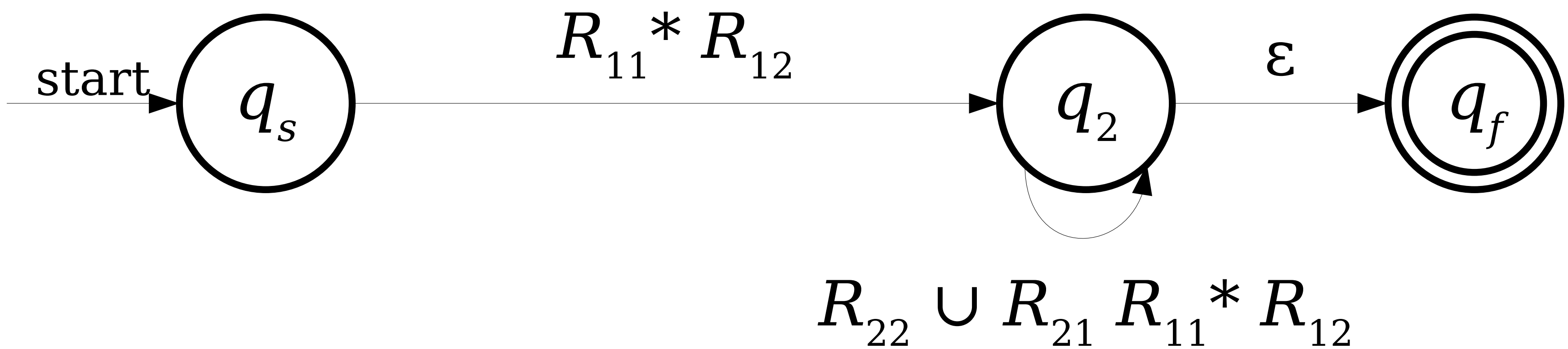


From NFAs to Regular Expressions

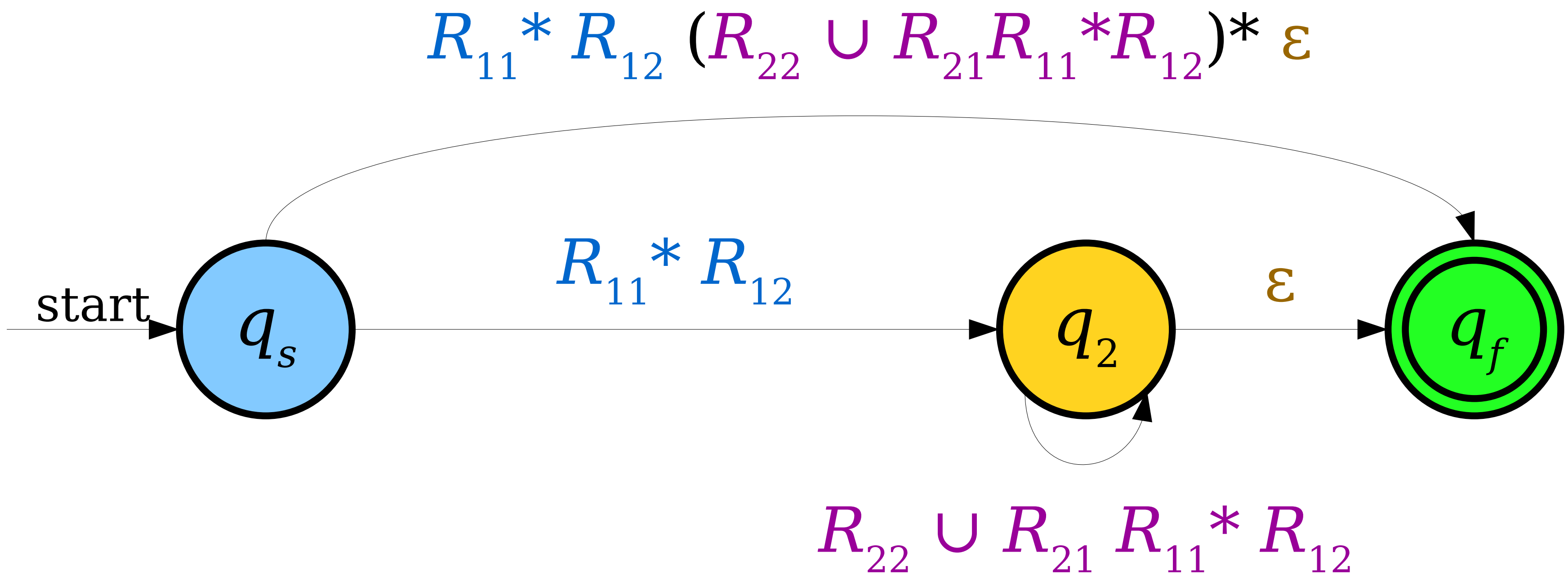


Note: We're using union to combine these transitions together.

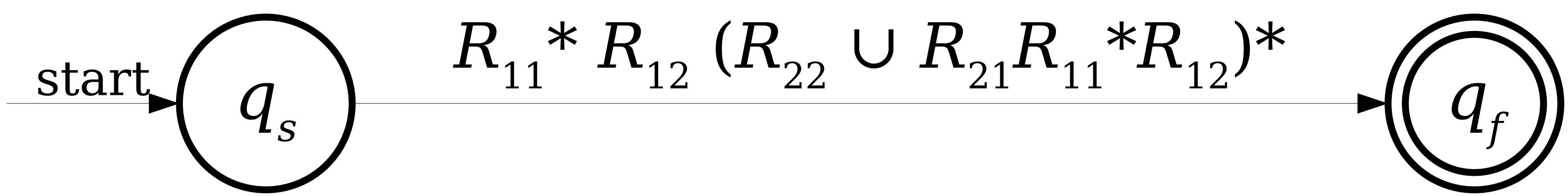
From NFAs to Regular Expressions

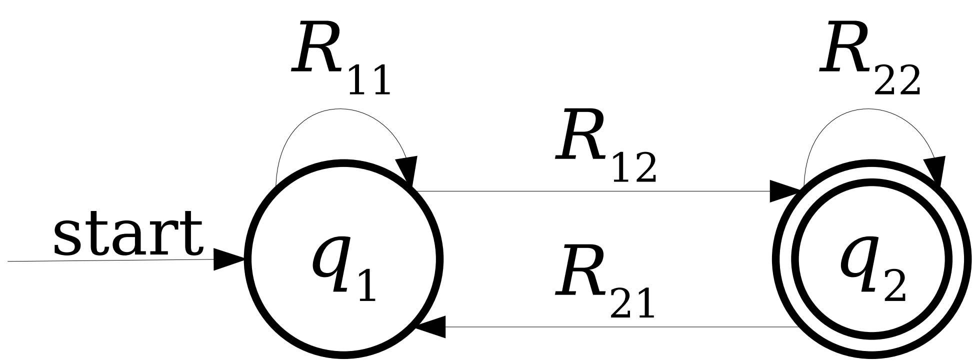


From NFAs to Regular Expressions

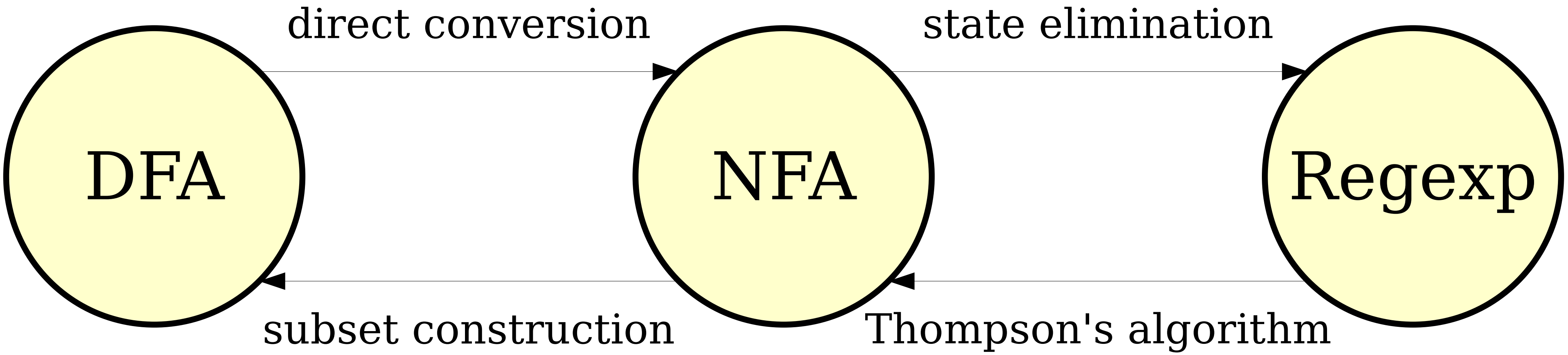


From NFAs to Regular Expressions





Our Transformations



Theorem: The following are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathscr{L}(D) = L$.
- There is an NFA N such that $\mathscr{L}(N) = L$.
- There is a regular expression R such that $\mathscr{L}(R) = L$.

Why This Matters

- The equivalence of regular expressions and finite automata has practical relevance.
  - Regular expression matchers have all the power available to them of DFAs and NFAs.
- This also is hugely theoretically significant: the regular languages can be assembled “from scratch” using a small number of operations!

Your Action Items

- ***Read "Guide to Regexes"***
  - There's a lot of information and advice there about how to write regular expressions, plus a bunch of worked exercises.
- ***Read "Guide to State Elimination"***
  - It's a beautiful algorithm. The Guide goes into a lot more detail than what we did here.

Next Time

- ***Intuiting Regular Languages***
  - What makes a language regular?
- ***The Myhill-Nerode Theorem***
  - The limits of regular languages.